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"Almost" Mean-Field Ising Model: An Algebraic Approach

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We study the thermodynamic limit of the algebraic dynamics for an "almost" mean-field Ising model, which is a slight generalization of the Ising model in the mean-field approximation. We prove that there exists a family of „relevant" states on which the algebraic dynamics α^t can be defined. This α^t defines a group of automorphisms of the algebra obtained by completing the standard spin algebra with respect to the quasiuniform topology defined by our states.

KEY WORDS: Spin systems; algebraic approach; thermodynamical limit.

1. INTRODUCTION

In recent years great effort has been made to include in the algebraic formulation of quantum systems introduced by Haag and Kastler[(1)] a larger and larger number of models describing physical phenomena.

However, several models have been shown not to fit into this algebraic setup.

This is the case of long-ranged spin systems: for them, in fact, Robinson's constraint on the potential is not satisfied and therefore the dynamics *cannot* be defined as a norm limit of the infrared cutoff dynamics α^t_V.[(2)] With regard to continuous systems, even if no rigorous result exists on the subject, it is quite clear that for long-range interactions (LRI) the time evolution of a local variable involves sequences of delocalized operators whose norm-convergence for $V \to \infty$ cannot be stated in general (see ref. 3 for concrete models, providing counterexamples).

There are two possible ways, not mutually exclusive, to approach the problem.

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On one hand, one can select a certain family of "relevant states" where the dynamics can be defined (this is the typical approach proposed by Dubin and Sewell[4] and further developed by several authors. See, for instance, ref. 5 and references therein).

On the other hand, one can try to "enlarge" the algebraic setup, to allow also unbounded observables (or even more general objects) to be included therein.

In this spirit several algebraic structures have been introduced and extensively studied in recent years (mostly from the mathematical point of view): *-algebras of unbounded operators, in brief O^*-algebras (see ref. 6 for an overview), quasi-*-algebras,[7,8] partial *-algebras,[9] and CQ^*-algebras.[10]

In particular, the problem of performing rigorosly the thermodynamic limit of some local observables was the starting point for the introduction of quasi *-algebras (see discussion in ref. 7): the basic idea was, in fact, to complete the algebra of local observables in a suitably chosen topology so as to include thermodynamic limits (the completion of a locally convex *-algebra provides, in fact, the most typical instance of a quasi-*-algebra).

In ref. 7 this formalism has been applied to the spin model describing the BCS model of superconductivity in Anderson's language. However, in our opinion, the use of this framework seems not to be essential for mean-field spin models, even though it allows one to recover a purely algebraic solution for the removal of the infrared cutoff in the equation of motion.

This aspect has been discussed by Bagarello and Morchio,[12] where the same algebraic results as in ref. 7 are obtained without making use of O^*-algebras. This is essentially due to the fact that all variables appearing in the equation of motion are uniformly bounded with respect to the volume V. This can be seen by considering $\|\sigma_\alpha^V\|$ with

$$\sigma_\alpha^V = \frac{1}{|V|} \sum_{i \in V} \sigma_\alpha^i, \qquad \alpha = 1, 2, 3 \tag{1.1}$$

(Of course σ_α^V is the relevant variable in performing the infinite-volume limit.)

In this paper we will discuss the spin model described by the finite-volume Hamiltonian

$$H_V = \frac{J}{|V|^\gamma} \sum_{ij \in V} \sigma_3^i \sigma_3^j = J |V|^{1-\gamma} \sigma_3^V \sum_{i \in V} \sigma_3^i \tag{1.2}$$

with $0 < \gamma \leqslant 1$, where the sum is extended to all the lattice sites in the volume V. This is, for $\gamma = 1$, a typical mean-field Ising model that can be studied using the same techniques as ref. 11.

There a method to perform the infinite-volume limit of the algebraic dynamics α_V^t has been introduced. The basic idea is to exploit the analytical dependence of α_V^t on σ_α^V. This allows the infinite-volume limit to pass through the analytical function defining α_V^t in terms of t, α_α^k, σ_β^V. The fact that this function does not depend explicitly on V makes it easy to perform the limit, which exists in the ultrastrong sense with respect to a family of "relevant" states.

These analyticity techniques seem not to be immediately adaptable to the model described by (1.2), due to the nonuniform boundness of the V-dependent operators appearing in the equation of motion.

Of course, since the model is highly long-ranged, we have little chance of finding results about the existence of the thermodynamic limit of α_V^t without making reference to a family of states. This corresponds to the physical fact that not all states are "relevant" in the sense of ref. 4, i.e., the infinite-volume limit of α_V^t cannot be performed on all the states over the standard spin algebra A_S. In particular, for mean-field spin models such states need to be regular enough to ensure the convergence of $\boldsymbol{\sigma}^V$ in the ultrastrong topology induced by them. This is, in other words, a condition of sufficient regularity of the states at large distances; see ref. 11.

With regard to the physical relevance of our model, we recall that Hamiltonians depending on $V^{-1/2}$ have been studied in ref. 12.

2. INTRODUCTION TO THE MODEL

The finite-volume Hamiltonian H_V which describes our model is

$$H_v = \frac{J}{|V|^\gamma} \sum_{ij \in V} \sigma_3^i \sigma_3^j = J|V|^{1-\gamma} \sigma_3^V \sum_{i \in V} \sigma_3^i \tag{2.1}$$

The Heisenberg equations of motion are given by

$$\frac{d}{dt} \alpha_V^t(\sigma_\alpha^k) = i[H_V, \alpha_V^t(\sigma_\alpha^k)], \qquad \alpha = 1, 2, 3$$

which gives

$$\begin{aligned}\frac{d}{dt} \alpha_V^t(\sigma_\alpha^k) &= -2 \frac{J}{|V|^\gamma} \varepsilon_{3\alpha\beta} \sum_{i \in V} [\sigma_3^i, \alpha_V^t(\sigma_\beta^k)] \\ &= -2J|V|^{1-\gamma} \varepsilon_{3\alpha\beta} \{\sigma_3^v, \alpha_V^t(\sigma_\beta^k)\}\end{aligned} \tag{2.2}$$

It is clear that the dependence of the rhs on its variables is analytical entire. We can solve the above equation, using the algebra of the Pauli matrices. The solution is, for $\alpha = 1, 2, 3$,

$$\begin{aligned}\alpha_V^t(\alpha_\alpha^k) &= \exp(iH_V t)\,\sigma_\alpha^k \exp(-iH_V t)\\ &= \sigma_\alpha^k \cos^2(S_3^V) - 2\varepsilon_{3\alpha\beta}\sigma_\beta^k \sin(S_3^V)\cos(S_3^V)\\ &\quad + \sigma_3^k\sigma_\alpha^k\sigma_3^k \sin^2(S_3^V) + O(|V|^{-\gamma})\end{aligned} \tag{2.3}$$

where the following quantities have been defined:

$$S_3^V = 2\frac{J}{|V|^\gamma} t \sum_{i\in V} \sigma_3^i = 2Jt\,|V|^{1-\gamma}\sigma_3^V \tag{2.4}$$

and

$$\sigma_3^V = \frac{1}{|V|}\sum_{i\in V}\sigma_3^i \tag{2.5}$$

We claim that terms of the $|V|^{-\gamma}$ order will not play any role in the convergence discussion since they are norm converging to zero. Therefore we will neglect such corrections in the following.

The problem we consider whether the infinite-volume limit of α_v^t makes sense under appropriate conditions.

The relevant points are the following:

1. First of all we see explicitly from Eqs. (2.4) and (2.5) that there is no uniform boundedness with respect to V for the variable S_3^V; therefore the framework discussed in ref. 7 seems to assume a crucial relevance.

2. The alternative method developed in ref. 11 cannot be applied any longer. In fact, Eqs. (2.2) and (2.3) explicitly depend on the volume V; whose $(1-\gamma)$th power enters in the equation of motion; therefore the condition of having a small V dependence in the rhs of the equation of motion is of course not satisfied. This means that the dependence on V cannot be handled as a small perturbation going to zero uniformly.

3. The third point is related to refs. 5 and 7, which clearly show that the current way of studying the thermodynamic limit of α_V^t for systems with LRI is to select a family of states. We will show in the next section that the "relevant" states for our Ising model are the antiferromagnetic states in the z direction, or local modifications of such states.

We now state the general result without going into the details, which will all be discussed in the next section:

It is possible to introduce a family F of states which define, via the

GNS construction, representations in which the infinite, volume limit of α_V^t exists in an appropriate topology t_0. This limit is a group of one-parameter automorphisms of the completion $A = \tilde{A}_S(\xi_0)$ of the standard spin C^*-algebra.

3. MATHEMATICAL PROOFS AND DETAILS

3.1. Notations and Basic Definitions

Let D be a pre-Hilbert space; by $L^+(D)$ we will denote the *-algebra of all closable operators A defined on D such that $AD \subseteq D$, $A^*D \subseteq D$.

Following ref. 6, we refer to any *-subalgebra of $L^+(D)$ as an O^*-algebra.

Let M be a self-adjoint operator in Hilbert space H; then $D = D^\infty(M) = \bigcap_{k>1} D(M^k)$ is the natural domain for the polynomial algebra generated by M. This kind of domain occurs very frequently in applications. The space $D^\infty(M)$ is a reflexive Fréchet domain with respect to the topology defined by the seminorms

$$f \in D \to \|f\|_n = \|M^n f\|; \qquad n \in N \tag{3.1}$$

The space $L^+(D)$, for $D = D^\infty(M)$, can be made a locally convex *-algebra if we define a topology τ by the set of seminorms

$$A \in L^+(D) \to \|A\|^{f,k} = \max\{\|M^k A f(M)\|, \|f(M) A M^k\|\} \tag{3.2}$$

where f runs over the set C of all continuous, bounded, nonnegative functions on $(0, \infty)$ decreasing faster than any inverse power and $\|\cdot\|$ denotes the usual C^*-norm of bounded operators in H.

The *-algebra $L^+(D)$ is complete under the topology τ, which is often referred to as the "quasi-uniform topology."[7]

Now let A be an (abstract)*-algebra and J a set of indices. For $i \in J$, let π_i be a faithful *-representation of A on D_i, dense domain of the Hilbert space H_i; i.e., π_i is a *-homomorphism of A into $L^+(D_i)$.

Let us assume that $D_i = D^\infty(M_i)$ for some self-adjoint operator M_i in H_i. We denote by τ_i the corresponding topology on $L^+(D_i)$ defined by (3.2).

Then by A we can define a topology ξ_0, called a "physical topology" by Lassner,[7] in the following way: by means of π_i, A is identified with on O^*-algebra, $\pi_i(A)$, on D_i; then we can consider on A the topology, which we still call τ_i, induced by the τ_i of $L^+(D_i)$:

$$\|A\|_i^{f,k} = \|\pi_i(A)\|^{f,k} \tag{3.3}$$

The topology ξ_0 is, then, the supremum of the τ_i.

In ref. 7 it has been shown that $A[\xi_0]$ is a locally convex *-algebra whose completion $\tilde{A}[\xi_0]$ is still an algebra. This is a nontrivial result which depends on the facts (a) that A is naturally embedded in each $L^+(D_i)$ and (b) each $L^+(D_i)$ can be considered as an O^*-algebra on the space

$$D_J = \sum_{i \in J} \oplus D_i \tag{3.4}$$

In fact, each element (A_i) of $\prod_i L^+(D_i)$ acts as an operator on D_J.

Of course the family $\{\pi_i\}$ of representations, defining the topology ξ_0, can be built up starting from a family ω_i of faithful states on A, via the well-known GNS construction for arbitrary *-algebras.[6]

This is just what we will do in the next section: from a family $\{|\{\mathbf{n}\}\rangle\}$ of states, we will define a *-representation $\pi_{\{n\}}$ on certain spaces $D_{\{n\}}$ of the form $D_{\{n\}} = D^\infty(M_{\{n\}})$, of the local *-algebra of spin operator A_S and we will define the topology ξ_0 as described above. The completion of the union of all these algebras will play for us the role of our observable algebra.

3.2. General Theorems and Definitions

Throughout this section we will adopt the same notations as in ref. 7 and 13.

Let $H_\infty = \bigotimes_p C_p^2$ be the infinite tensor product of the 2-dimensional spaces C^2; see ref. 14.

We call $|\mathbf{n}\rangle$ the unit vector in C^2, which is characterized by the condition $(\boldsymbol{\sigma}\mathbf{n})|\mathbf{n}\rangle = |\mathbf{n}\rangle$. This determines $|\mathbf{n}\rangle$ up to a phase factor. The scalar product of two such vectors is given by

$$\langle \mathbf{n} | \mathbf{n}' \rangle = e^{i\varphi}[\tfrac{1}{2}(1 + \mathbf{n} \cdot \mathbf{n}')]^{1/2}$$

Let $\{\mathbf{n}\} = \{\mathbf{n}_1, \mathbf{n}_2, ...\}$; then

$$|\{\mathbf{n}\}\rangle = \bigotimes_p |\mathbf{n}_p\rangle \tag{3.5}$$

denote unit vectors in H_∞. Further let $H_{\{n\}}$ be the separable Hilbert space generated by all vectors $|\{\mathbf{n}'\}\rangle$ which are equivalent to $|\{\mathbf{n}\}\rangle$ in the sense of ref. 13. One can choose a special basis in $H_{\{n\}}$, which one gets from $|\{\mathbf{n}\}\rangle$ by flipping a finite number of spins. For this one chooses two three-vectors $\mathbf{n}^1$, $\mathbf{n}^2$, which together with $\mathbf{n}$ form an orthonormal basis and put $\mathbf{n}^\pm = 1/2(\mathbf{n}^1 \pm \mathbf{n}^2)$. Then $(\boldsymbol{\sigma}\mathbf{n}^+)|\mathbf{n}\rangle = 0$ and if we set $|m, n\rangle = (\boldsymbol{\sigma}\mathbf{n}^-)^m |\mathbf{n}\rangle$, then

$$(\boldsymbol{\sigma} \cdot \mathbf{n})|m, \mathbf{n}\rangle = (-1)^m |m, \mathbf{n}\rangle, \qquad m = 0, 1 \tag{3.6}$$

Now $|\{m\}, \{\mathbf{n}\}\rangle = \bigotimes_p |m_p, \mathbf{n}_p\rangle$, $m_p = 0, 1$, $\sum m_p < \infty$, form a countable orthonormal basis in $H_{\{n\}}$. In this space we define the unbounded self-adjoint operator M (see ref. 7), by

$$M|\{m\}, \{\mathbf{n}\}\rangle = \left(1 + \sum_p m_p\right) |\{m\}, \{\mathbf{n}\}\rangle \tag{3.7}$$

The operator $M - 1$ counts the number of flipped spins with respect to the "ground" state $|\{0\}, \{\mathbf{n}\}\rangle$.

Of course $M = M_{\{n\}}$ depends on $\{\mathbf{n}\}$, but we will drop this dependence whenever it is clear from the context in what space it acts.

Now let

(i) $F_{\{n\}} = \bigcap_k D(M^k)$

(ii) $\tau_{\{n\}}$ be the quasiuniform topology on $L^+(D_{\{n\}})$

(iii) $\pi_{\{n\}}: A_S \to L^+(D_{\{n\}})$ be the natural realization (representation) of A_S on $D_{\{n\}}$ defined by

$$\pi_{\{n\}}(\sigma_\alpha^p)|\{m\}, \{\mathbf{n}\}\rangle = \sigma_\alpha^p |m_p, \mathbf{n}_p\rangle \otimes \left(\prod_{p' \neq p} \otimes |m_{p'} \mathbf{n}_{p'}\rangle\right) \tag{3.8}$$

These representations are faithful since A_S is a simple C^*-algebra.

We choose a fixed ordering of the lattice points p, and therefore denote them simply by natural numbers $p = 1, 2, 3,\ldots$.

We define a set F of vectors which generalizes the family Σ defined in ref. 7 and allows one even to extend the "relevant states" introduced in ref. 11:

$$F = \left\{(\mathbf{n}_p\}: \lim_{V \to \infty} \frac{1}{|V|^\gamma} \sum_{p=1}^{|V|} \mathbf{n}_p = \eta \mathbf{n},\ |\mathbf{n}| = 1, 0 \leqslant \eta \leqslant 1, \mathbf{n}_p = (0, 0, \pm 1)\right\} \tag{3.9}$$

where we have also $\mathbf{n} = (0, 0, \pm 1)$ and $\eta \mathbf{n}$ represents the "almost" mean value of the sequence $\{\mathbf{n}_p\}$.

We see that for $\gamma = 1$ this family almost coincides with the ones introduced in the quoted papers. The physical interpretation of (3.9) is essentially the following: the F-states are the usual up–down states typical of antiferromagnetic matter, or their local modification pointing in the z direction. This is of course in essential agreement with our physical picture of an antiferromagnet.

The topology ξ_0 we will use is defined starting from the $\tau_{\{n\}}$ through the system of seminorms:

$$\|A\|_{\{n\}}^{f,k} = \max\{\|M_{\{n\}}^k \pi_{\{n\}}(A) f(M_{\{n\}})\|, \|f(M_{\{n\}}) \pi_{\{n\}}(A) M_{\{n\}}^k\|\} \tag{3.10}$$

where $f \in C$, $k = 0, 1, 2,\ldots$, and $\{\mathbf{n}\} \in F$.

We observed that the M operator can be represented by σ-matrices, i.e.,

$$M = 1 + \tfrac{1}{2} \sum_p [1 - (\boldsymbol{\sigma}_p \cdot \mathbf{n}_p)] \tag{3.11}$$

where $\{\mathbf{n}_p\} \in F$ and the series on the rhs converges strongly because it is the supremum of an increasing net of positive operators.

We are now ready to prove the following result.

Lemma 1. The quantity S_3^V defined in Eq. (2.4) has a limit S_3 in the ξ_0-topology defined by the seminorms (3.10). This limit is defined in the completion $A = \tilde{A}_S[\xi_0]$, which is a topological *-algebra. Moreover, even $(S_3^V)^n$ converges in the same topology to $(S_3)^n$, $\forall n = 0, 1, \ldots$.

Proof. In order to prove the first part of the lemma, it is sufficient to prove that S_3^V converges with respect to every seminorm $\|A\|^{f,k}$ in every space $D_{\{n\}}$, $\{\mathbf{n}\} \in F$. [We write here and in the following for simplicity $M = M_{\{n\}}$, $A = \pi_{\{n\}}(A)$.] To further simplify the notation, we will consider only the first contribution in (3.2), so that $\|A\|^{f,k}$ will be essentially identified with the single $\|M^k A f(M)\|$. This is allowed by the fact that $\|f(M)\, A M^k\| = \|M^k A^* f(M)\|$, for f real, so that its estimate for A^* is the same as $\|M^k A f(M)\|$.

Let

$$M = \sum_{m=1}^{\infty} m P_m \tag{3.12}$$

be the spectral decomposition of M. Then we can write[7]

$$\|A\|^{f,k} = \sum_{lm} l^k \|A\|_{lm} f(m) \tag{3.13}$$

where

$$\|A\|_{l,m} = \|P_l A P_m\| \tag{3.14}$$

From its definition it is easy to see that we can write

$$\mathbf{S}^V = 2 \frac{J}{|V|^\gamma} t \sum_{p=1}^{|V|} \mathbf{n}_p (\boldsymbol{\sigma}_p \cdot \mathbf{n}_p) = (0, 0, S_3^V) \tag{3.15}$$

where $\mathbf{n}_p = (0, 0, \pm 1)$.

Threfore the action of $\mathbf{S}^V$ on the vectors of $D_{\{n\}}$, $|\{m\}, \{\mathbf{n}\}\rangle$, can be determined using Eq. (3.6). We get

$$\mathbf{S}^V |\{m\}, \{\mathbf{n}\}\rangle = 2 \frac{J}{|V|^\gamma} t \sum_{p=1}^{|V|} \mathbf{n}_p (-1)^{m_p} |\{m\}, \{\mathbf{n}\}\rangle \tag{3.16}$$

Taking into account that the Hilbert space $H_{\{m\}}$ is such that $1+\sum m_p = m$, we get

$$\|S_3^V - 2Jt\eta n_3\|_{l,m} \leqslant \delta_{lm} \sup_{[\{m_p\}:1+\sum m_p = m]} |2Jt| \left| \frac{1}{|V|^\gamma} \sum_{p=1}^{|V|} n_3^p (-1)^{m_p} - \eta n_3 \right|$$
$$\leqslant \delta_{lm} |2Jt| \left\{ \left| \frac{1}{|V|^\gamma} \sum_{p=1}^{|V|} n_3^p - \eta n_3 \right| + \frac{2m}{|V|^\gamma} \right\} \tag{3.17}$$

and there, recalling (3.13),

$$\|S_3^V - 2Jt\eta n_3\|^{f,k} \leqslant |2Jt| \left| \frac{1}{|V|^\gamma} \sum_{p=1}^{|V|} n_3^p - \eta n_3 \right| \sum_m m^k f(m)$$
$$+ \frac{|4Jt|}{|V|^\gamma} \sum_m m^{k+1} f(m) \to 0 \tag{3.18}$$

for $V \to \infty$.

In (3.18) we have used the fact that the function f is in the class C and therefore the sum in the above inequality is finite. In (3.17) estimates like the following have been used:

$$\left| \frac{1}{|V|^\gamma} \sum_{p=1}^{|V|} n_3^p (-1)^{m_p} - \eta n_3 \right|$$
$$= \left| \frac{1}{|V|^\gamma} (-n_3^1 - \cdots - n_3^{m-1} + n_3^m + \cdots + n_3^{|V|}) - \eta n_3 \right|$$
$$\leqslant \left| \frac{1}{|V|^\gamma} \sum_{p=1}^{|V|} n_3^p - \eta n_3 \right| + \frac{2m}{|V|^\gamma}$$

(This is the particular case where $m_1 = m_2 = \cdots = m_{m-1} = 1$, $m_m = m_{m+1} = \cdots = 0$, but it also can be adapted to any other sequence $\{m_p\}$ such that $1+\sum m_p = m$.)

The algebraic nature of A has been discussed in Section 3.1. We have only to prove now that also the powers of S_3^V are convergent. We note that for any integer n the following inequality holds:

$$\|(S_3^V)^n - (2Jt\eta n_3)^n\|_{l,m}$$
$$= \left\| \sum_{k=1}^{n} \binom{n}{k} (S_3^V - 2Jt\eta n_3)^k (2Jt\eta n_3)^{n-k} \right\|_{l,m}$$
$$\leqslant \sum_{k=1}^{n} \binom{n}{k} (\|S_3^V - 2Jt\eta n_3\|_{l,m})^k \, |2Jt\eta|^{n-k} \tag{3.19}$$

and therefore an estimate like the one in (3.18) holds, so that $\|(S_3^V)^n - (2Jt\eta n_3)^n\|^{f,k} \to 0$ for V going to inifinity.

In deriving this result, we took into account that, despite the fact that $\|\cdots\|_{l,m}$ is not a Banach norm, the equality $\|(\cdots)^k\|_{l,m} = (\|\cdots\|_{l,m})^k$ holds since $S_3^V - 2Jt\eta n_3$ commutes with M and therefore with its spectral projections.

Finally we notice that $[S_3, \sigma_\alpha^i] = 0$, $\forall \alpha, i$.

The above result allows us to prove the following:

Proposition 2. The finite-volume dynamics α_V^t converges in the topology ξ_0, uniformly for t running in a compact set, to an automorphism α^t of the subalgebra of A, A^G, generated by A_S and by the continuous functions of S_3.

Proof. We use the explicit solution of the equation of motion, formula (2.3), in order to estimate the following quantity:

$$\begin{aligned}\|\alpha_V^t(\sigma_\alpha^i) - \alpha_{V'}^t(\sigma_\alpha^i)\|^{f,k} \leqslant\; & \|\sigma_\alpha^i[\cos^2(ts_3^V) - \cos^2(ts_3^{V'})]^{f,k} \\ & + \|\varepsilon_{3\alpha\beta}\sigma_\beta^i[\sin(2ts_3^V) - \sin(2ts_3^{V'})]\|^{f,k} \\ & + \|\sigma_3^i\sigma_\alpha^i\sigma_3^i[\sin^2(ts_3^V) - \sin^2(ts_3^{V'})]\|^{f,k}\end{aligned}$$

where $s_3^V = S_3^V/t = 2J|V|^{1-\gamma}\sigma_3^V$.

We put now $s_3^{V'} = s_3^V + s_3^{\Delta V}$, $\Delta V = V' - V$, and observe that, since s_3^V converges in the $\|\cdots\|^{f,k}$ norm, $\|s_3^{\Delta V}\|^{f,k} < \varepsilon$ for V and V' big enough. Moreover, from ref. 7 we have

$$\|M^k\sigma_\alpha^i M^{-k}\| \leqslant c_{\sigma_\alpha}$$

which allows one to derive the following inequality:

$$\|\alpha_V^t(\sigma_\alpha^i) - \alpha_{V'}^t(\sigma_\alpha^i)\|^{f,k} \leqslant 2(c_{\sigma_\alpha} + c_{\sigma_\beta})\|\sin(ts_3^{\Delta V})\|^{f,k}$$

Taking now into account that, if $A \leqslant B$ and if $[A, M] = [B, M] = 0$, then $\|A\|^{f,k} \leqslant \|B\|^{f,k}$ and that $\sin(A) \leqslant A$, we can conclude that

$$\|\alpha_V^t(\sigma_\alpha^i) - \alpha_{V'}^t(\sigma_\alpha^i)\|^{f,k} \leqslant 2t(c_{\sigma_\alpha} + c_{\sigma_\beta})\|s_3^{\Delta V}\|^{f,k} \to 0$$

for $V, V' \to \infty$, and uniformly for t running in a compact set.

Finally, due to the completeness of A in the ξ_0-topology we conclude that there exists $\alpha^t(\sigma_\alpha^i) \in A$ such that $\alpha^t(\sigma_\alpha^i) = \xi_0\text{-}\lim_{V\to\infty} \alpha_V^t(\sigma_\alpha^i)$. Actually it is easy to deduce from Eq. (2.3) that $\alpha^t(\sigma_\alpha^i)$ belongs to A^G, and that α^t maps A^G into itself since $\alpha^t(s_3) = s_3$. The automorphism nature of α^t follows from the fact that $\|\alpha_V^t(A)\,\alpha_V^t(B) - \alpha^t(A)\,\alpha^t(B)\|^{f,k}$ can be estimated to go to zero for $V \to \infty$ with similar techniques as before.

We have just proved that the algebraic dynamics can be defined with a limiting procedure. Now we are interested in finding the equation of motion satisfied by α^t. This is the main content of Proposition 3. The same proposition allows us to prove the group nature of α^t, but not to extend the dynamics to the whole algebra A. This will be achieved in Corollary 4 by introducing a self-adjoint operator H_{eff} which gives the right time evolution of any observables.

Proposition 3. The algebraic dynamics α^t defined in Proposition 2 satisfies the infinite-volume limit of the equation of motion (2.2) and is a group of automorphisms of A^G.

Proof. Equation (2.2) can be written in the form

$$\frac{d}{dt}\alpha_V^t(\sigma_\alpha^k) = F(\alpha_V^t(\sigma_\alpha^k), S_3^V)$$

where F is a polynomial in its variables.

We want to perform the ξ_0-limit for $V \to \infty$ of the above equation.

First, we observe that the finite-volume solution of the equation of motion (2.2), Eq. (2.3), can be inserted in the function $F(\alpha_V^t(\sigma_\alpha^k), S_3^V)$ defining in this way a new function $F^\#$:

$$F^\#(t, \sigma_\alpha^k, s_3^V) = F(\alpha_V^t(\sigma_\alpha^k), S_3^V)$$

which puts in evidence the time dependence of the function $F^\#$. Therefore the finite-volume equation of motion above can be rewritten in the more convenient form

$$\frac{d}{dt}\alpha_V^t(\sigma_\alpha^k) = F^\#(t, \sigma_\alpha^k, s_3^V) \tag{3.20}$$

Using the same kind of estimate as in Proposition 2, we can prove that $F^\#$ is ξ_0-continuous with respect to its variables and therefore

$$\begin{aligned}\xi_0\text{-}\lim_{V\to\infty} F^\#(t, \sigma_\alpha^k, s_3^V) &= F^\#(t, \sigma_\alpha^k, \xi_0\text{-}\lim_{V\to\infty} s_3^V)\\ &= F^\#(t, \sigma_\alpha^k, s_3)\end{aligned}$$

where obviously $s_3 = S_3/t$.

For the same reason $F^\#$ satisfies the Lipschitz condition in t (which is the relevant variable for our purpose). This means that

$$\|F^\#(t, \sigma_\alpha^k, s_3^V) - F^\#(s, \sigma_\alpha^k, s_3^V)\|^{f,k} \leqslant M\,|t-s|$$

where M is the Lipschitz constant.

We will use the previous estimate to prove the existence of the infinite-volume limit of the lhs of (3.20). First of all we notice that the time derivative in (3.20) is defined in a natural way as

$$\frac{d}{dt}\alpha_V^t(\sigma_\alpha^k)=\xi_0\text{-}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\alpha_V^{t+\varepsilon}(\sigma_\alpha^k)-\alpha_V^t(\sigma_\alpha^k)\right] \tag{3.21}$$

We will prove that

$$\xi_0\text{-}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\alpha_V^{t+\varepsilon}(\sigma_\alpha^k)-\alpha_V^t(\sigma_\alpha^k)\right]$$

is uniform in V, so that the following result holds:

$$\xi_0\text{-}\lim_{V\to\infty}\frac{d}{dt}\alpha_V^t(\sigma_\alpha^k)=\frac{d}{dt}\xi_0\text{-}\lim_{V\to\infty}\alpha^t{}_V(\sigma_\alpha^k)=\frac{d}{dt}\alpha^t(\sigma_\alpha^k)$$

interchanging the infinite-volume limit with the time derivative.

The final result is therefore that the infrared cutoff in Eq. (3.20) can be removed and the following differential equation is obtained:

$$\frac{d}{dt}\alpha^t(\sigma_\alpha^k)=F^{\#}(t,\sigma_\alpha^k,s_3) \tag{3.22}$$

The proof of the uniformity is simple but a little bit long. We only give the essential steps. We have to study the following quantity:

$$\left\|\frac{1}{\varepsilon}\left[\alpha_V^{t+\varepsilon}(\sigma_\alpha^k)-\alpha_V^t(\sigma_\alpha^k)\right]-\frac{1}{\varepsilon'}\left[\alpha_V^{t+\varepsilon'}(\sigma_\alpha^k)-\alpha_V^t(\sigma_\alpha^k)\right]\right\|^{f,k}$$

We first write Eq. (3.20) in integral form:

$$\alpha_V^{t'}(\sigma_\alpha^k)=\alpha_V^t(\sigma_\alpha^k)+\int_t^{t'}F^{\#}(s,\sigma_\alpha^k,s_3^V)\,ds,\qquad \forall t'>t$$

We can now substitute this equation for $t'=t+\varepsilon$ and for $t'=t+\varepsilon'$ in the $\|\cdots\|^{f,k}$ above. Further we define a function G by

$$G(t,s;\sigma_\alpha^k,s_3^V)=F^{\#}(s,\sigma_\alpha^k,s_3^V)-F^{\#}(t,\sigma_\alpha^k,s_3^V)$$

and substitute $F^{\#}(s,\sigma_\alpha^k,s_3^V)$ in the integrals defining $\alpha_V^{t'}(\sigma_\alpha^k)$. Of course $F^{\#}(t,\sigma_\alpha^k,s_3^V)$ is constant in s and we can integrate it. Using the Lipschitz condition, we get after a little computation

$$\left\|\frac{1}{\varepsilon}\left[\alpha_V^{t+\varepsilon}(\sigma_\alpha^k)-\alpha_V^t(\sigma_\alpha^k)\right]-\frac{1}{\varepsilon'}\left[\alpha_V^{t+\varepsilon'}(\sigma_\alpha^k)-\alpha_V^t(\sigma_\alpha^k)\right]\right\|^{f,k}\leqslant\frac{M}{2}(\varepsilon+\varepsilon')$$

which shows the uniformity in the volume of

$$\xi_0\text{-}\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [\alpha_V^{t+\varepsilon}(\sigma_\alpha^k) - \alpha_V^t(\sigma_\alpha^k)]$$

and allows us to conclude.

With regard to the group property, we start by observing that $\alpha^t(\sigma_\alpha^k)$ can be written in the form

$$\alpha^t(\sigma_\alpha^k) = \exp(its_3\sigma_3^k)\, \sigma_\alpha^k \exp(-its_3\sigma_3^k)$$

This implies that $\alpha^{t+\tau}(\sigma_\alpha^k) = \alpha^t[\alpha^\tau(\sigma_\alpha^k)]$. Moreover, due to the automorphism property of α^t, $\alpha^t(\sigma_\alpha^k \sigma_\beta^l) = \alpha^t(\sigma_\alpha^k)\, \alpha^t(\sigma_\beta^l)$, one can extend the group property to the whole A^G.

Finally, we prove the following results.

Corollary 4. The time evolution α^t can be extended to an automorphisms (denoted by the same symbol) of the whole algebra A and it is a group of automorphisms of A.

Proof. We only need to prove that α^t is continuous from $A_S[\xi_0]$ into $A[\xi_0]$, so that it can be extended to the whole $A[\xi_0]$.

The proof makes use of the possibility, extensively discussed in refs. 13 and 15, of introducing an effective Hamiltonian H_{eff} giving rise to the same equation of motion obtained from Eq. (2.3) after taking the thermodynamic limit. It is easy to see that such an effective Hamiltonian is

$$H_{\text{eff}} = 2J\eta n_3 \sum_p [\pi_n(\sigma_3^p) - n_3^p]$$

where n_3 is the z component of the $\mathbf{n}$-vector defining the family F and the summation is extended to the whole lattice. In fact, minor manipulations show that

$$\alpha_V^t(\sigma_\alpha^k) = \exp(iH_V t)\, \sigma_\alpha^k \exp(-iH_V t)$$

defined in (2.3) converges in the topology ξ_0 to

$$\alpha^t(\sigma_\alpha^k) = \exp(iH_{\text{eff}} t)\, \sigma_\alpha^k \exp(-iH_{\text{eff}} t)$$

analogous to what is required in ref. 13 for the BCS model. H_{eff} turns out to be a well-defined self-adjoint operator in each representation space $H_{\{n\}}$. Therefore we have

$$\begin{aligned}\|\alpha^t(A)\|^{f,k} &= \|M^k \exp(iH_{\text{eff}} t)\, A \exp(-iH_{\text{eff}} t)\, f(M)\| \\ &= \|\exp(iH_{\text{eff}} t)\, M^k A f(M) \exp(-iH_{\text{eff}} t)\| \\ &= \|M^k A f(M)\| = \|A\|^{f,k}\end{aligned}$$

since M and H_{eff} commute. This proves the continuity of α' and therefore the corollary.

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